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On geometric distance-regular graphs with diameter three

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ABSTRACT

In this paper we study distance-regular graphs with intersection array

$$\{(t+1)s, ts, (t-1)(s+1-\psi); 1, 2, (t+1)\psi\} \quad (1)$$

where s, t, ψ are integers satisfying $t \geq 2$ and $1 \leq \psi \leq s$. Geometric distance-regular graphs with diameter three and $c_2 = 2$ have such an intersection array. We first show that if a distance-regular graph with intersection array (1) exists, then s is bounded above by a function in t. Using this we show that for a fixed integer $t \geq 2$, there are only finitely many distance-regular graphs of order (s, t) with smallest eigenvalue $-t-1$, diameter $D = 3$ and intersection number $c_2 = 2$ except for Hamming graphs with diameter three. Moreover, we will show that if a distance-regular graph with intersection array (1) for $t = 2$ exists then $(s, \psi) = (15, 9)$. As Gavrilyuk and Makhnev (2013) [9] proved that the case $(s, \psi) = (15, 9)$ does not exist, this enables us to finish the classification of geometric distance-regular graphs with smallest eigenvalue -3, diameter $D \geq 3$ and $c_2 \geq 2$ which was started by the first author (Bang, 2013) [1].

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1. Introduction

For unexplained definitions and notations the reader is referred to Section 2. Recall that a non-complete distance-regular graph Γ with valency k and smallest eigenvalue $\theta_{\min}$ is called *geometric* if there exists a set $\mathcal{C}$ of cliques such that each edge lies in exactly one clique in $\mathcal{C}$ and each clique in $\mathcal{C}$ is a Delsarte clique, i.e., a clique of exactly $1 + k/(-\theta_{\min})$ vertices (see [10]). So a geometric distance-regular graph is the point graph of a partial linear space where the set of lines is a set of Delsarte cliques. It was shown in [13] that for a positive integer $m \geq 2$ there are only finitely many coconnected distance-regular graphs with valency at least three and smallest eigenvalue at least $-m$ that are not geometric.

In this paper we study geometric distance-regular graphs with diameter three. A geometric distance-regular graph with diameter three has intersection array

$$\{(t+1)s, t(s+1-\psi_1), (t-t_2)(s+1-\psi_2); 1, (t_2+1)\psi_1, (t+1)\psi_2\}, \tag{2}$$

where $0 \leq t_2 < t$ and $1 \leq \psi_1 \leq \psi_2 \leq s$ are all integers (see [13, Lemma 4.1]). Examples of geometric distance-regular graphs with diameter three are the Hamming graph $H(3, q)$, the Johnson graph $J(n, 3)$ $n \geq 6$, the Grassmann graph $J_q(n, 3)$ $n \geq 6$, the bilinear forms graph $H_q(n, 3)$ and so on. Note that not every distance-regular graph with intersection array (2) is geometric. For example, the Doob graph with diameter three (i.e., the Cartesian product of a Shrikhande graph with a complete graph on 4 vertices), which is not geometric, has the same intersection array as the Hamming graph $H(3, 4)$.

A distance-regular graph is exactly a generalized hexagon if and only if it has intersection array (2) with $t_2 = 0$ and $\psi_1 = \psi_2 = 1$. A regular near hexagon is exactly a graph with intersection array (2) and $\psi_1 = \psi_2 = 1$ that is locally the disjoint union of cliques. For larger diameter, the regular near $2D$-gons of order (s, t) with $D \geq 4$, $c_2 \geq 3$ and $s \geq 2$ are exactly dual polar graphs, see [8, Theorem 9.11]. The case satisfying $D \geq 4$, $c_2 = 2$ and $s \geq 2$ is still open. Regular near hexagons are even less known, see [7] for some recent progress. In this paper we will consider graphs with intersection array (2) satisfying $c_2 = 2$ and $s \geq 1$. In this case it follows by [13, Lemma 4.2 (i)] that $t_2 = 1$ and $\psi_1 = 1$.

For positive integers s, t, ψ satisfying $t \geq 2$ and $\psi \leq s$, we denote by $G(s, t; \psi)$ a distance-regular graph with intersection array

$$\{(t+1)s, ts, (t-1)(s+1-\psi); 1, 2, (t+1)\psi\}. \tag{3}$$

Although a graph $G(s, t; \psi)$ is not necessarily geometric the following lemma shows that it usually is.

Lemma 1.1. *Let $\Gamma = G(s, t; \psi)$ be a distance-regular graph with intersection array (3), where s, t, ψ are integers satisfying $t \geq 2$ and $1 \leq \psi \leq s$. If parameters s and t satisfy*

$$s > \begin{cases} 2(t+1)^2 + 1 & \text{if } t \geq 3 \\ 6 & \text{if } t = 2 \end{cases}$$

then Γ is geometric with smallest eigenvalue $-t-1$.

Proof. Note that $-t-1$ is the smallest eigenvalue of Γ. If parameters s and t satisfy $t \geq 3$ and $s > 2(t+1)^2 + 1$, then Γ is geometric with smallest eigenvalue $-t-1$ by [13, Theorem 5.3]. If $s > 6$ and $t = 2$, then the result immediately follows by [1, Theorem 3.1]. ■

It was shown in [11, Corollary 2] that for a thick regular near $2D$-gon with order (s, t), the number t is bounded by a function in s and D, i.e., $t < s^{\frac{4D}{h}} - 1$ where $h := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. With the same proof, it can be shown that this bound also holds for geometric distance-regular graphs. We will show in the next theorem that if Γ is a $G(s, t; \psi)$ then s is bounded by a function in t, which gives us a dual result to the result of Hiraki and Koolen [11, Corollary 2].

Theorem 1.2. *Let an integer $t \geq 2$ be given. Then there exists a positive constant $C := C(t)$ (only depending on t) such that if a graph $G(s, t; \psi)$ exists where s, ψ are integers satisfying $1 \leq \psi \leq s$ and $(t, \psi) \neq (2, 1)$ then*

$$s \leq C$$

holds (and hence $\psi \leq C$).

To prove this result we show that usually a graph $G(s, t; \psi)$ has only integral eigenvalues (see Lemma 3.2). This situation is similar to the case of regular near hexagons. It was shown by Shad and Shult [15] that a regular near hexagon has integral spectrum unless it is a generalized hexagon. However, the graph $G(1, 5; 1)$ with intersection array $\{5, 4, 3; 1, 2, 5\}$, which arises as the point-block incidence graph of the square 2-(11, 5, 2)-design, has irrational eigenvalues $\pm\sqrt{3}$.

It is known that there are no geometric distance-regular graphs with smallest eigenvalue -2, diameter $D \geq 3$ and $c_2 \geq 2$ (see [6, Theorem 3.12.2, Theorem 4.2.16]). Bang [1, Theorem 4.3] has shown that any geometric distance-regular graph Γ with smallest eigenvalue -3, diameter $D \geq 3$ and $c_2 \geq 2$ satisfies one of the following:

(a) The Hamming graph $H(3, s+1)$, where $s \geq 2$.
(b) The Johnson graph $J(s-1, 3)$, where $s \geq 7$.
(c) The collinearity graph of the generalized quadrangle of order $(s, 3)$ deleting the edges in a spread, where $s \in \{3, 5\}$.
(d) $\Gamma = G(s, 2; \psi)$ with intersection array $\{3s, 2s, s+1-\psi; 1, 2, 3\psi\}$, where $1 < \psi < s$.

We will show in Theorem 1.3 that if a graph $G(s, 2; \psi)$ exists, where s and ψ are integers with $1 < \psi < s$, then $(s, \psi) = (15, 9)$.

Theorem 1.3. *For any given integers s and ψ with $1 < \psi < s$, if a distance-regular graph with intersection array $\{3s, 2s, s+1-\psi; 1, 2, 3\psi\}$ does exist then $(s, \psi) = (15, 9)$.*

As Gavrilyuk and Makhnev [9] proved that a $G(15, 2; 9)$ (with intersection array $\{45, 30, 7; 1, 2, 27\}$) does not exist, we have the following result.

Theorem 1.4. *A geometric distance-regular graph with smallest eigenvalue -3, diameter $D \geq 3$ and $c_2 \geq 2$ is one of the following.*

(i) *The Hamming graph $H(3, s+1)$, where $s \geq 2$.*
(ii) *The Johnson graph $J(s-1, 3)$, where $s \geq 7$.*
(iii) *The collinearity graph of the generalized quadrangle of order $(s, 3)$ deleting the edges in a spread, where $s \in \{3, 5\}$.*

In Section 3, we prove Theorem 1.2. To show this result we consider the two cases, $\psi > \frac{t}{2(t+1)}s$ and $\psi \leq \frac{t}{2(t+1)}s$. If $\psi > \frac{t}{2(t+1)}s$ then we prove Lemma 3.1 by showing that the multiplicity of the smallest eigenvalue of the corresponding dual graph is bounded above by a function in t. In this case s is also bounded above by a function in t. On the other hand, if $\psi \leq \frac{t}{2(t+1)}s$ then we prove in Lemma 3.2 that there exists a finite set S such that if $(s, \psi) \notin S$ then any graph $G(s, t; \psi)$ has only integral eigenvalues. Using Theorem 1.2, we show in Theorem 3.4 that for a fixed integer $t \geq 2$, there are only finitely many distance-regular graphs of order (s, t) with smallest eigenvalue $-t-1$, diameter $D = 3$ and intersection number $c_2 = 2$ except for the Hamming graphs with diameter three. In Section 4, we prove Theorem 1.3 by showing in Lemma 4.1 that $\psi \leq \frac{1}{3}s$ does not occur.

2. Preliminaries

All the graphs considered in this paper are finite, undirected and simple. The reader is referred to [6] for more background information. For a connected graph Γ, the distance $d(x, y)$ between two vertices x, y of Γ is the length of a shortest path between x and y in Γ, and the diameter D is the maximum distance between any two vertices of Γ. Let $V(\Gamma)$ be the vertex set of Γ. For any vertex $x \in V(\Gamma)$, let $\Gamma_i(x)$ be the set of vertices in Γ at distance precisely i from x, where i is a non-negative integer not exceeding D. The *adjacency matrix* A_Γ of a graph Γ is the $(|V(\Gamma)| \times |V(\Gamma)|)$-matrix with rows and columns indexed by $V(\Gamma)$, where the (x, y)-entry of A_Γ equals 1 whenever $d(x, y) = 1$ and 0 otherwise. The *eigenvalues* of Γ are the eigenvalues of A_Γ. Let $\theta_0, \theta_1, \ldots, \theta_n$ be the distinct eigenvalues of Γ and let $m_\Gamma(\theta_i)$ be the *multiplicity* of $\theta_i (i = 0, 1, \ldots, n)$. A sequence of vertices $W = w_0, w_1, \ldots, w_\ell$, which are not necessarily mutually distinct, is called a *walk* of length ℓ if w_i and w_{i+1} are adjacent for each $i = 0, \ldots, \ell - 1$. The number of walks of length ℓ from x to y is given by $(A_\Gamma^\ell)_{(x,y)}$, where $(A_\Gamma^\ell)_{(x,y)}$ is the (x, y)-entry of matrix A_Γ^ℓ. If $w_0 = w_\ell$ then W is called a *closed walk*. Let $Tr(A_\Gamma^\ell)$ denote the trace of A_Γ^ℓ (i.e., the sum of the diagonal entries of A_Γ^ℓ). Then we have

$$\sum_{i=0}^{n} m_\Gamma(\theta_i)\theta_i^\ell = Tr(A_\Gamma^\ell) = \text{the number of closed walks of length } \ell \text{ in } \Gamma \quad (\ell \geq 1). \tag{4}$$

A connected graph Γ is called a *distance-regular graph* if there exist integers $b_i, c_i, i = 0, 1, \ldots, D$, such that for any two vertices x, y at distance $i = d(x, y)$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$ where D is the diameter of Γ. In particular, Γ is regular with valency $k := b_0$. The numbers b_i, c_i and $a_i := k - b_i - c_i$ $(0 \leq i \leq D)$ are called the *intersection numbers* of Γ. Set $c_0 = b_D = 0$. We observe $a_0 = 0$ and $c_1 = 1$. Array

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$$

is called the *intersection array* of Γ. We define $k_i := |\Gamma_i(x)|$ for any vertex x and $i = 0, 1, \ldots, D$. Then we have

$$k_0 = 1, \qquad k_1 = b_0, \qquad k_{i+1} = \frac{k_i b_i}{c_{i+1}} \quad (i = 0, 1, \ldots, D - 1). \tag{5}$$

Suppose that Γ is a distance-regular graph with valency $k \geq 2$ and diameter $D \geq 2$. It is well known that Γ has exactly $D + 1$ distinct eigenvalues which are the eigenvalues of the following tridiagonal matrix

$$L_1(\Gamma) := \begin{pmatrix} 0 & b_0 & & & & & \\ c_1 & a_1 & b_1 & & & & \\ & c_2 & a_2 & b_2 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & c_i & a_i & b_i & \\ & & & & \cdot & \cdot & \cdot \\ & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & c_D & a_D \end{pmatrix} \tag{6}$$

(cf. [6, p.128]). The *standard sequence* $(u_i(\theta))_{0 \leq i \leq D}$ corresponding to an eigenvalue θ of Γ is a sequence satisfying the following recurrence relation:

$$u_0(\theta) = 1, \qquad u_1(\theta) = \frac{\theta}{k}, \qquad c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) = \theta u_i(\theta) \quad (1 \leq i \leq D).$$

Then the multiplicity of the eigenvalue θ is given by

$$m_\Gamma(\theta) = \frac{|V(\Gamma)|}{\sum_{i=0}^{D} k_i u_i^2(\theta)}, \tag{7}$$

which is known as *Biggs' formula* (cf. [4, Theorem 21.4], [6, p.128]). Let $\mathbb{N}$ denote the set of positive integers. Recall that the *local graph* of a vertex x is the subgraph of Γ induced by the set of neighbors of x in Γ, and a *clique* is a set of pairwise adjacent vertices. A distance-regular graph is of *order* (s, t) if the local graph of any vertex is the disjoint union of $t + 1$ cliques of size s for some positive integers s, t. A distance-regular graph of order (s, t) is called a *regular near* $2D$*-gon of order* (s, t) if $a_i = c_i(s - 1)$ $(i = 1, 2, \ldots, D)$.

3. Proof of Theorem 1.2

In this section we will show Theorem 1.2, which implies that for any given integer $t \geq 2$ there exists a positive constant $C := C(t)$ such that if $s > C$ and a graph $G(s, t; \psi)$ exists then $(t, \psi) = (2, 1)$. To show Theorem 1.2 we consider the two cases, $\psi > \frac{t}{2(t+1)}s$ and $\psi \leq \frac{t}{2(t+1)}s$. If $\psi > \frac{t}{2(t+1)}s$ then we prove Lemma 3.1 by showing that the multiplicity of the smallest eigenvalue of the corresponding dual graph is bounded above by a function in t. In this case s is bounded above by a function in t. On the other hand, if $\psi \leq \frac{t}{2(t+1)}s$ then we prove in Lemma 3.2 that there exists a finite set S such that if $(s, \psi) \notin S$ then any graph $G(s, t; \psi)$ has only integral eigenvalues.

For given integers s, t, ψ with $t \geq 2$ and $1 \leq \psi \leq s$, let $\Gamma := G(s, t; \psi)$. By (6), Γ has exactly four distinct eigenvalues $\theta_0 > \theta_1 > \theta_2 > \theta_3$:

$$\theta_0 = (t+1)s, \qquad \theta_i = \frac{3s - 2\psi - 1 + (-1)^{i-1}\sqrt{(s+1-2\psi)^2 + 4(t-1)s}}{2} \quad (i = 1, 2), \qquad \theta_3 = -t - 1. \tag{8}$$

By (5) and (7), we find

$$|V(\Gamma)| = \frac{(s+1)\left\{(t^2 - t)s^2 + 2\psi(st+1)\right\}}{2\psi} \tag{9}$$

and

$$m_\Gamma(\theta_3) = m_\Gamma(-t-1) = \frac{s^2(s+1-\psi)\left\{(t^2-t)s^2 + 2\psi(st+1)\right\}}{\psi(2s^2 + 2s - 2\psi s + 2st + t^2 + t - 2\psi t)}. \tag{10}$$

Suppose that Γ is geometric. The *dual graph* of Γ, denoted by $\widehat{\Gamma}$, is the graph whose vertices are the Delsarte cliques of Γ (i.e., cliques of size $s+1$) and two Delsarte cliques are adjacent if they intersect. Let B be the vertex–(Delsarte clique) incidence matrix (i.e., the (0, 1)-matrix with rows and columns indexed by the vertex set and the set of Delsarte cliques respectively, where the (x, C)-entry of B is 1 if the vertex x is contained in the Delsarte clique C and 0 otherwise). Then

$$BB^T = A_\Gamma + (t+1)I_{|V(\Gamma)|} \quad \text{and} \quad B^TB = (s+1)I_{|V(\widehat{\Gamma})|} + A_{\widehat{\Gamma}}, \tag{11}$$

where B^T is the transpose of B and I_v is the $v \times v$ identity matrix. By double-counting the number of ones in B, we find

$$|V(\widehat{\Gamma})|(s+1) = |V(\Gamma)|(t+1) \tag{12}$$

and thus

$$|V(\widehat{\Gamma})| = (t+1)(st+1) + \frac{t(t^2-1)s^2}{2\psi}. \tag{13}$$

In particular, $\widehat{\Gamma}$ is a regular graph with valency $t(s+1)$.

Lemma 3.1. *Let an integer $t \geq 2$ be given. Then there exists a positive constant $C := C(t)$ (only depending on t) such that if a graph $G(s, t; \psi)$ exists where s, ψ are integers satisfying $1 \leq \psi \leq s$ and $\frac{t}{2(t+1)}s < \psi \leq s$ then*

$$s \leq C$$

holds.

Moreover if $t = 2$ then either $s \leq 6$ or $(s, \psi) = (15, 9)$ holds.

Proof. Let s and ψ be positive integers satisfying $\frac{t}{2(t+1)}s < \psi \leq s$, and let $\Gamma := G(s, t; \psi)$. To prove this lemma, it is enough to show that s is bounded above by a function only depending on t. If $s \leq 2(t+1)^2 + 1$ then the result follows immediately. We now assume $s > 2(t+1)^2 + 1$. Then Γ is geometric by Lemma 1.1. Moreover, we have $|V(\widehat{\Gamma})| < |V(\Gamma)|$ by (12) and $s > 2(t+1)^2 + 1 > t$, and hence 0 is an eigenvalue of BB^T. First assume that B^TB is invertible. Then the multiplicity of eigenvalue 0 for the matrix BB^T is $|V(\Gamma)| - |V(\widehat{\Gamma})|$, which is also equal to $m_\Gamma(-t-1)$ by $BB^T = A_\Gamma + (t+1)I_{|V(\Gamma)|}$ in (11). By $m_\Gamma(-t-1) = |V(\Gamma)| - |V(\widehat{\Gamma})|$, (9), (10) and (13), we find

$$\psi = \frac{(s+t)(t+1)}{2t}. \tag{14}$$

Substituting (14) in (13), we find

$$|V(\widehat{\Gamma})| - \left\{(t+1)(st+1) + st^2(t-1) - t^3(t-1)\right\} = \frac{t^4(t-1)}{s+t} \tag{15}$$

whose both sides are positive integers as $t > 1$. This shows that if B^TB is invertible then we obtain $s \leq t^4(t-1) - t$.

Now assume that the matrix B^TB is singular. Then 0 is an eigenvalue of both BB^T and B^TB, and thus $-s-1$ is an eigenvalue of $A_{\widehat{\Gamma}}$ by $B^TB = (s+1)I_{|V(\widehat{\Gamma})|} + A_{\widehat{\Gamma}}$ in (11). As B^TB is positive semidefinite, by (11), we find that $-s-1$ is the smallest eigenvalue of the dual graph $\widehat{\Gamma}$ with multiplicity

$$m_{\widehat{\Gamma}}(-s-1) = |V(\widehat{\Gamma})| - |V(\Gamma)| + m_\Gamma(-t-1). \tag{16}$$

Since the dual graph $\widehat{\Gamma}$ of Γ is a regular graph with valency $t(s+1)$ and smallest eigenvalue $-s-1$, it follows by (4) that

$$\begin{aligned} t(s+1)|V(\widehat{\Gamma})| = Tr(A_{\widehat{\Gamma}}^2) = \sum_{\eta:\text{ eigenvalue of }\widehat{\Gamma}} m_{\widehat{\Gamma}}(\eta)\eta^2 \geq t^2(s+1)^2 \\ + m_{\widehat{\Gamma}}(-s-1)(-s-1)^2. \end{aligned} \tag{17}$$

Since we have $|V(\widehat{\Gamma})| < (t^3 + 2t^2 - 1)s + t + 1$ from (13) and the condition $\psi > \frac{t}{2(t+1)}s$,

$$1 \leq p := m_{\widehat{\Gamma}}(-s-1) \leq \frac{t\left\{|V(\widehat{\Gamma})| - t(s+1)\right\}}{s+1} < t^4 + 2t^3 - t^2 - t \tag{18}$$

follows by (17). Hence $1 \leq p < t^4 + 2t^3 - t^2 - t$. By (9), (10), (13) and (16),

$$\begin{aligned} p &= m_{\widehat{\Gamma}}(-s-1) = |V(\widehat{\Gamma})| - |V(\Gamma)| + m_\Gamma(-t-1) \\ &= \frac{t(s + t(s+1-2\psi+t))(2\psi(st+1) + (t^2-t)s^2)}{2\psi(2s^2 + 2s - 2\psi(s+t) + t(2s+t+1))}. \end{aligned} \tag{19}$$

By (13) and (19), we have

$$\begin{aligned} &-\frac{t^2(t^2-1)s^2}{2\psi} - t - t^2(s+1) - 2(s-\psi)t^3 + (s+1)t^4 - t^5 + 2(s+1-\psi)p \\ &= \frac{(2t^4 - 2t^2)\psi - (t^2-t)(t^4+p)}{s+t} \end{aligned} \tag{20}$$

where both sides are integers. If $(2t^4 - 2t^2)\psi - (t^2-t)(t^4+p) = 0$ then by

$$2(t+1)^2 + 1 < s < \frac{2(t+1)\psi}{t} = \frac{t^4+p}{t^2}$$

it follows that $p > t^4 + 4t^3 + 3t^2$, a contradiction to (18). Hence the following number q is a non-zero integer, where

$$q := \frac{(2t^4 - 2t^2)\psi - (t^2 - t)(t^4 + p)}{s + t} \quad \text{and thus}$$
$$s = \left(\frac{2t^4 - 2t^2}{q}\right)\psi - \frac{(t^2 - t)(t^4 + p)}{q} - t. \tag{21}$$

By $s > t$, (21), $\psi > \frac{t}{2(t+1)}s$ and $p < t^4 + 2t^3 - t^2 - t$, we have

$$2sq > (s + t)q = (2t^4 - 2t^2)\psi - (t^2 - t)(t^4 + p)$$
$$> st^3(t - 1) - (t^2 - t)(2t^4 + 2t^3 - t^2 - t)$$

and thus

$$q > \frac{t^3(t - 1)}{2} - \frac{(t^2 - t)(2t^4 + 2t^3 - t^2 - t)}{2s} > \frac{t^3(t - 1)}{2}$$
$$- \frac{(t^2 - t)(2t^4 + 2t^3 - t^2 - t)}{2t} > -t^5. \tag{22}$$

It follows by (21) and (22) that $-t^5 < q < 2t^4 - 2t^2$ as $\psi \le s < s + t$ and p is a positive integer. Substituting s of (21) to (20), we obtain a non-zero polynomial in ψ of degree at most three with coefficients as functions in p, q and t. Hence, it follows by $1 \le p < t^4+2t^3-t^2-t$, $-t^5 < q < 2t^4-2t^2$ and (21) that s is bounded above by a function $C(t)$ which is dependent on t.

Now we consider the case $t = 2$. Suppose $s > 6$. Then Γ is geometric by Lemma 1.1. As we find $|V(\widehat{\Gamma})| < |V(\Gamma)|$ by (12) with $s > t = 2$, BB^T is singular. If B^TB is invertible then parameters s and ψ satisfy $(s, \psi) = (14, 12)$ as $\psi = \frac{3(s+2)}{4} \in \mathbb{N}$ and $\frac{16}{s+2} \in \mathbb{N}$ (see (14) and (15)). If $(s, \psi) = (14, 12)$ then θ_1 and θ_2 are irrationals and thus

$$m_\Gamma(\theta_1) = m_\Gamma(\theta_2) = \frac{|V(\Gamma)| - m_\Gamma(\theta_0) - m_\Gamma(\theta_3)}{2} = \frac{135}{2} \notin \mathbb{N},$$

which is impossible. Hence B^TB is singular. It follows by (18), (21) and (22) that the following are all integers

$$p := m_{\widehat{\Gamma}}(-s - 1) \quad \text{and} \quad q := \frac{24\psi - 2(p + 16)}{s + 2} \tag{23}$$

with $1 \le p \le 25$ and $-17 < q \le 23$. Now we will show $24\psi - 2(p + 16) \ne 0$ (i.e., $q \ne 0$). If $24\psi - 2(p + 16) = 0$ then we find $\psi = 3$, $p = 20$ and $s \in \{7, 8\}$ as $6 < s < 3\psi = \frac{p+16}{4} \le \frac{41}{4}$. Then it follows by (9), (10), (13) and (23) that $|V(\Gamma)| = \frac{(s+1)(s^2+6s+3)}{3}$, $m_\Gamma(-3) = \frac{s^2(s-2)(s^2+6s+3)}{3(s^2-3)}$, $|V(\widehat{\Gamma})| = s^2 + 6s + 3$ and $m_{\widehat{\Gamma}}(-s - 1) = p = 20$. But they do not satisfy (16). Thus $24\psi - 2(p + 16) \ne 0$ (i.e., $q \ne 0$). Now substituting $s = \frac{24\psi-2(p+16)-2q}{q}$ of (23) in (20), we find

$$2\left\{(p - 8)q^2 - 24(p - 2)q + 1728\right\}\psi^2 + \left\{q^3 + 2(p + 7)q^2 + 4(p^2 + 14p - 176)q\right.$$
$$\left. - 576(p + 16)\right\}\psi + 24(p + q + 16)^2 = 0. \tag{24}$$

For any integers $1 \le p \le 25$ and $-17 < q \le 23$, there exists the unique pair $(s, \psi) = (15, 9)$ satisfying $\frac{1}{3}s < \psi \le s$, (23) and (24). This shows that if $s > 6$ then $(s, \psi) = (15, 9)$, which completes the proof. ■

The incidence graph of the 2-(11, 5, 2) design (with intersection array {5, 4, 3; 1, 2, 5}) has irrational eigenvalues $\pm\sqrt{3}$. On the other hand, all the eigenvalues of the regular near hexagon (with intersection array {24, 22, 20; 1, 2, 12}) are integers. In Lemma 3.2 we will show that for a fixed integer $t \ge 2$ there exists a finite set $S(t)$ such that if integers s and ψ satisfy both $1 \le \psi \le \frac{t}{2(t+1)}s$ and

$(s, \psi) \notin S(t)$ then any graph $G(s, t; \psi)$ has only integral eigenvalues. Using Lemma 3.2 we can easily show that regular near hexagons with $c_2 = 2$ and $s \geq 3$ have only integral eigenvalues since if $\psi = 1$ then the set $S(t)$ in Eq. (25) is $\{(1, 1)\}$.

Given an integer $t \geq 2$, define a set

$$S(t) := \left\{ (s, \psi) \in \mathbb{N} \times \mathbb{N} \mid F(s, \psi) = 0, \ \ \psi \in \left\{ 1, 2, \ldots, \left\lfloor \frac{2 + \sqrt{t^2 - t + 4}}{2} \right\rfloor \right\} \right\}, \tag{25}$$

where

$$\begin{aligned} F(s, \psi) := {} & 2(t-1)s^3 + (\psi(-6t + 10) + 3t^2 - 5t + 2)s^2 \\ & + (4\psi^2(t-4) - 2\psi(t^2 - 3t - 2) - t^2 + t)s \\ & + 2\psi(4\psi^2 - 2\psi(t+2) + t + 1). \end{aligned} \tag{26}$$

For each integer ψ satisfying $1 \leq \psi \leq \left\lfloor \frac{2+\sqrt{t^2-t+4}}{2} \right\rfloor$, $F(s, \psi)$ is a non-zero polynomial in s of degree 3, and hence $|S(t)| \leq 3 \left\lfloor \frac{2+\sqrt{t^2-t+4}}{2} \right\rfloor$.

Lemma 3.2. *Let an integer $t \geq 2$ be given. If a graph $G(s, t; \psi)$ has a non-integral eigenvalue, where s, ψ are integers satisfying $1 \leq \psi \leq \frac{t}{2(t+1)} s$ then*

$$(s, \psi) \in S(t)$$

holds, where $S(t)$ is the finite set defined in (25).

Proof. Let $t \geq 2$ be an integer. For given integers s and ψ satisfying $1 \leq \psi \leq \frac{t}{2(t+1)} s$, let $\Gamma := G(s, t; \psi)$. Assume that Γ has a non-integral eigenvalue. Then θ_1 and θ_2 in (8) must be irrational numbers, and the equation $Tr(A_\Gamma) = \sum_{i=0}^{3} m_\Gamma(\theta_i)\theta_i = 0$ implies $m_\Gamma(\theta_1) = m_\Gamma(\theta_2)$ and thus

$$m_\Gamma(\theta_1) = m_\Gamma(\theta_2) = \frac{(t+1)(m_\Gamma(\theta_3) - s)}{3s - 2\psi - 1} = \frac{|V(\Gamma)| - 1 - m_\Gamma(\theta_3)}{2} \tag{27}$$

follows by (8) and $|V(\Gamma)| = \sum_{i=0}^{3} m_\Gamma(\theta_i)$. By substituting (9) and (10) in (27), we find that s and ψ must satisfy the equation $F(s, \psi) = 0$, see (26). To complete the proof, we need to show $1 \leq \psi \leq \left\lfloor \frac{2+\sqrt{t^2-t+4}}{2} \right\rfloor$ (i.e., $(s, \psi) \in S$). We first show the following claim.

Claim 3.3. *Suppose $F(s, \psi) = 0$. If $\frac{1}{2}(2 + \sqrt{t^2 - t + 4}) < \psi \leq \frac{t}{2(t+1)} s$ then $s < 2\psi$.*

Proof of Claim 3.3. Suppose $\psi > \frac{1}{2}(2 + \sqrt{t^2 - t + 4})$. As $\psi > \frac{1}{2}(2 + \sqrt{t^2 - t + 4}) > \frac{1}{2}(t + 1)$, $F(0, \psi) = 2\psi(2\psi - 1)(2\psi - (t + 1)) > 0$ and thus there is $s < 0$ satisfying $F(s, \psi) = 0$. As $F(2\psi, \psi) = 2\psi\{(4t^2 - 6t + 4)\psi - t^2 + 2t + 1\} > 0$ and the largest zero of the equation $\frac{\partial}{\partial s} F(s, \psi) = 0$ in s is

$$\frac{6\psi t - 3t^2 + 5t - 10\psi - 2 + \sqrt{(12t^2 + 4)\psi^2 + (-24t^3 + 72t^2 - 112t + 64)\psi + 9t^4 - 24t^3 + 25t^2 - 14t + 4}}{6(t-1)}$$

which is less than 2ψ, it follows that each real number s satisfying $F(s, \psi) = 0$ is less than 2ψ. This shows Claim 3.3. ■

As the condition $\psi \leq \frac{t}{2(t+1)} s$ implies $2\psi \leq \left(\frac{t}{t+1}\right) s < s$, we find by Claim 3.3 that if θ_1 and θ_2 are irrational numbers then $F(s, \psi) = 0$ holds and thus ψ must satisfy

$$\psi \leq \frac{1}{2}(2 + \sqrt{t^2 - t + 4}),$$

which shows $(s, \psi) \in S$. This completes the proof. ■

Using Lemmas 3.1 and 3.2 we now prove Theorem 1.2, which means that given an integer $t \geq 2$ there are only finitely many s's and ψ's such that a graph $G(s, t; \psi)$ exists with $(t, \psi) \neq (2, 1)$. It is known that a $G(s, 2; 1)$ with $s \geq 1$ is either the Hamming graph $H(3, s+1)$ or the Doob graph with diameter three (in this case $s = 3$), see [6, Corollary 9.2.5]. Since the Hamming graph $H(3, s+1)$ with $s \geq 1$ is a $G(s, 2; 1)$, it follows that for the pair $(t, \psi) = (2, 1)$ there are infinitely many s's such that a $G(s, 2; 1)$ exists.

Proof of Theorem 1.2. Let $t \geq 2$ be a given integer. Let s and ψ be integers such that $1 \leq \psi \leq s$ and $(t, \psi) \neq (2, 1)$. We want to show that there exists a positive constant $C = C(t)$ (only depending on t) such that if a graph $\Gamma = G(s, t; \psi)$ exists then $s \leq C$. We consider two cases, $\psi > \frac{t}{2(t+1)}s$ and $\psi \leq \frac{t}{2(t+1)}s$. In the first case the existence of the constant C follows from Lemma 3.1. In the case $\psi \leq \frac{t}{2(t+1)}s$, let $S = S(t)$ be the set as defined in (25). To complete the proof for given $t \geq 2$ and $(s, \psi) \notin S$ satisfying $1 \leq \psi \leq s$, $(t, \psi) \neq (2, 1)$ and $\psi \leq \frac{t}{2(t+1)}s$, we will show that s is bounded above by a function in t. It follows by Lemma 3.2 that if $(s, \psi) \notin S$ then both θ_1 and θ_2 are integers and thus $\sqrt{(s+1-2\psi)^2 + 4(t-1)s} = \theta_1 - \theta_2 = (s+1-2\psi) + r$, where r is a positive integer. As $\psi \leq \frac{t}{2(t+1)}s$ we find $1 \leq r < 2(t^2-1)$. It follows that

$$\psi = \left(\frac{r-2t+2}{2r}\right)s + \frac{r+2}{4} \tag{28}$$

where $1 \leq r < 2(t^2-1)$. Substituting (28) into (9) we find

$$\begin{aligned} &4(r-2t+2)^3\{|V(\Gamma)| - (s+1)(st+1)\} - r^3(r+2)^2(t^2-t) \\ &\quad - 2r(s+1)(t^2-t)(r-2t+2)\left\{2(r-2t+2)s - r^2 - 2r\right\} \\ &= \frac{r^3(r+2)^2(t^2-t)(r^2+4t-4)}{-(2r-4t+4)s - r(r+2)} \end{aligned} \tag{29}$$

where both sides are integers. Note here that $-(2r-4t+4)s - r(r+2) = -4r\psi \neq 0$ where the first equality follows from (28). If $2r-4t+4 \neq 0$ then $s \leq r^3(r+2)^2(t^2-t)(r^2+4t-4) + r(r+2) \leq f(t)$ holds as the absolute value of (29) is at least 1. If $2r-4t+4 = 0$, i.e., $r = 2(t-1)$, then $t = 2\psi$ by (28). Moreover, by (10),

$$\begin{aligned} &4m_\Gamma(\theta_3) - \left\{4(t-1)s^3 - 4t(t-2)s^2 + 2(t^2-1)(t-2)s - t(t^2-1)(t-2)\right\} \\ &= \frac{t^2(t-2)(t^2-1)}{2s+t} \end{aligned}$$

must be an integer. Since $t = 2$ implies $\psi = 1$, we have $t > 2$. Then there are only finitely many positive integers s such that $\frac{t^2(t-2)(t^2-1)}{2s+t}$ is an integer. Hence we showed that if $(s, \psi) \notin S$, $(t, \psi) \neq (2, 1)$ and $\psi \leq \frac{t}{2(t+1)}s$ both hold then s is bounded above by a certain function only depending on t. This completes the proof since S is a finite set with $|S| \leq \left\lfloor \frac{3(2+\sqrt{t^2-t+4})}{2} \right\rfloor$ and each s and ψ satisfying $(s, \psi) \in S$ are bounded above by a function on t from the definition of the set S (see (25)). ■

Mohar and Shawe-Taylor [14] (see also [6, Theorem 4.2.16]) characterized distance-regular graphs of order $(s, 1)$ with $s > 1$. The distance-regular graphs of order $(1, 2)$ and $(2, 2)$ were classified by Biggs, Boshier and Shawe-Taylor [5] and Hiraki, Nomura and Suzuki [12], respectively. Some strong results on distance-regular graphs of order $(s, 2)$ with $s > 2$ were given by Yamazaki [16]. In [2, Corollary 10.2], the authors showed that for a fixed integer $t > 1$, there are only finitely many distance-regular graphs of order (s, t) whose smallest eigenvalue is not equal to $-t-1$.

Using Theorem 1.2, we can show the following theorem.

Theorem 3.4. *For a fixed integer $t \geq 2$, there are only finitely many distance-regular graphs of order (s, t) with smallest eigenvalue $-t-1$, diameter $D = 3$ and intersection number $c_2 = 2$ except for Hamming graphs with diameter three.*

Proof. Let $t \geq 2$ be a given integer. Let Γ be a distance-regular graph of order (s, t) with smallest eigenvalue $-t-1$, diameter $D = 3$ and intersection number $c_2 = 2$. Then Γ is geometric with valency $b_0 = (t+1)s$. By [1, Lemma 4.1] (see also [3, Proposition 4.2 (i)]), the intersection numbers of Γ satisfy $b_i = (t+1-\tau_i)(s+1-\psi_i)$ $i = 1, 2$ and $c_j = \tau_j\psi_{j-1}$ $j = 1, 2, 3$, where parameters τ_i and ψ_i are as defined in [1, Section 4]. As any Delsarte clique in Γ has size $s+1 = a_1+2$, it follows by [3, Lemma 5.1 (i)] that $\psi_1 = 1$ which shows $\tau_2 = \tau_2\psi_1 = c_2 = 2$. Note here that Γ satisfies $\tau_1 = 1$ and $\tau_3 = t+1$ (see [1, Equation (9)]). Put $\psi := \psi_2$. Then Γ is a $G(s, t; \psi)$. If $s \neq 3$ then the condition $(t, \psi) = (2, 1)$ is equivalent to that Γ is the Hamming graph $H(3, s+1)$. As $b_0 = (t+1)s$ and $D = 3$, the result follows by Theorem 1.2. ■

In [13, Conjecture 7.5], the authors conjectured that for a fixed integer $t \geq 2$, any geometric distance-regular graph with smallest eigenvalue $-t-1$, diameter $D \geq 3$ and $c_2 \geq 2$ is either a Johnson graph, a Grassmann graph, a Hamming graph, a bilinear forms graph, or the number of vertices is bounded above by a function in t. Theorem 3.4 gives us more evidence that the conjecture is true.

4. Proof of Theorem 1.3

For given integers s and ψ with $1 \leq \psi \leq s$, let $\Gamma = G(s, 2; \psi)$. Then $\iota(\Gamma) = \{3s, 2s, s+1-\psi; 1, 2, 3\psi\}$. If $\psi = 1$ then $G(s, 2; \psi)$ is either the Hamming graph $H(3, s+1)$ or the Doob graph with diameter three (in this case $s = 3$), see [6, Corollary 9.2.5]. If $\psi = s$ then $G(s, 2; \psi)$ can be obtained as the collinearity graph of the generalized quadrangle of order $(s, 3)$ deleting the edges in a spread, where $s \in \{3, 5\}$ (see [1, Theorem 4.3]). In this section, we prove Theorem 1.3 which states that if a graph $G(s, 2; \psi)$ exists, where s and ψ are integers with $1 < \psi < s$ then $(s, \psi) = (15, 9)$. To prove Theorem 1.3, we need the following lemma.

Lemma 4.1. *Let s and ψ be any given integers with $1 < \psi < s$. If a graph $G(s, 2; \psi)$ exists then*

$$\psi > \frac{1}{3}s$$

holds.

Proof. Assume that a graph $\Gamma := G(s, 2; \psi)$ exists and $\psi \leq \frac{1}{3}s$. By Lemma 3.2 with $t = 2$, all the eigenvalues of Γ are integers as the set $S = \{(s, 2) \in \mathbb{N} \times \mathbb{N} \mid F(s, 2) = 0\}$ in (25) is empty. As $s+1-2\psi > 0$ holds from the assumption $\psi \leq \frac{1}{3}s$, we find by (8) that

$$\begin{aligned} &\theta_1 - \theta_2 = \sqrt{(s+1-2\psi)^2 + 4s} = (s+1-2\psi) + r \quad \text{and thus} \\ &\psi = \frac{(2r-4)s + r^2 + 2r}{4r} \end{aligned} \tag{30}$$

for some positive integer r. As $\psi = \frac{(2r-4)s+r^2+2r}{4r}$ is an integer with $1 < \psi \leq \frac{1}{3}s$, we find $r = 4$ and $s \geq 18$. Thus Γ is geometric by Lemma 1.1. Since the numbers $\psi = \frac{s+6}{4}$, $|V(\widehat{\Gamma})| = 18s - 69 + \frac{432}{s+6}$ and

$$\begin{aligned} &4(r-2)^3\{|V(\Gamma)| - (s+1)(2s+1)\} - 2r^3(r+2)^2 \\ &\quad - 4r(s+1)(r-2)\left\{2(r-2)s - r^2 - 2r\right\} \\ &= \frac{-2r^3(r+2)^2(r^2+4)}{(2r-4)s + r^2 + 2r} = \frac{-23040}{s+6} \end{aligned} \tag{31}$$

must be integers (see (9), (13) and (30)), s must satisfy

$$\frac{s+6}{4} \in \mathbb{N} \quad \text{and} \quad \frac{144}{s+6} \in \mathbb{N} \tag{32}$$

where $144 = \gcd(432, 23040)$. Since $s \geq 18$ also holds, we find by (32) that $s \in \{18, 30, 42, 66, 138\}$. But now $m_\Gamma(-3) = \frac{s(3s-2)(3s+2)(2s+3)}{(s+6)(3s+4)}$ is not a positive integer for any $s \in \{18, 30, 42, 66, 138\}$ (see (10)). Hence $\psi > \frac{1}{3}s$ follows. ■

Proof of Theorem 1.3. For any given integers s and ψ with $1 < \psi < s$, let $\Gamma := G(s, 2; \psi)$. As $\psi > \frac{1}{3}s$ holds by Lemma 4.1, it follows by Lemma 3.1 that either $s \leq 6$ or $(s, \psi) = (15, 9)$ holds. Since there are no integers $s \leq 6$ and ψ satisfying both $\frac{1}{3}s < \psi < s$ and $m_\Gamma(\theta_3) \in \mathbb{N}$ (see (10)), we find $(s, \psi) = (15, 9)$ which completes the proof. ■

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